Quantum Dynamics of Total Orders

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8 April 2020
QG Group Meeting

Main Questions

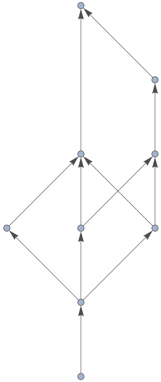
Can we understand quantum growth dynamics in terms of quantum random walks?

What basis states and transitions?

What does this tell us about fundamental dynamics?

Growing Discrete Spacetimes: Causal Set Approach

A causal set C_n is a set of n unlabeled elements $\{e_1, \ldots, e_n\}$ endowed with an irreflexive partial order relation $\prec$.



Discrete models are useful: they help us understand the partition function and path integral in QG

The continuum limit corresponds to $\ell \to 0$ – we often study the convergence to smooth Lorentzian structure

Growth models help us understand the problem of time, but is growth physical?

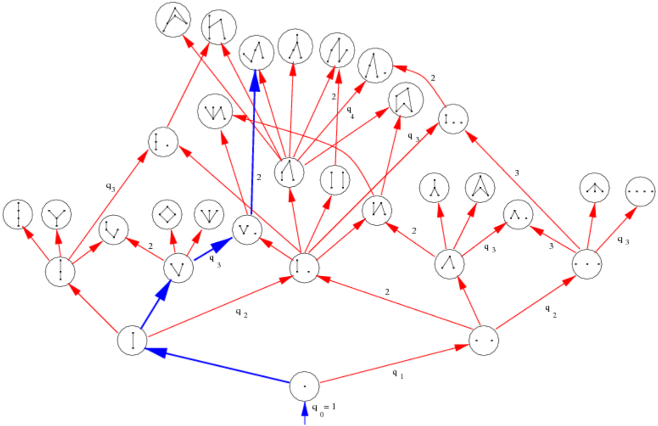
Can these models help us understand the fundamental dynamics between matter and geometry?

What are the most fundamental structures that can encode quantum growth dynamics?

Bombelli et al. '87
Surya '19

Ideas from Classical Growth Models

Classical Sequential Growth (CSG) categorizes discrete Lorentzian structures



The poset of finite causal sets

Rideout & Sorkin '99

Ideas from Classical Growth Models

We can formulate such a structure for any discrete Lorentzian theory such that:

- Building blocks are indistinguishable*
- There exists a unique, natural initial state
- A given state has a finite number of transitions to a state representing a larger spacetime
- The measure of an event does not depend on the growth sequence (discrete general covariance)
- Transitions follow the Kolmogorov sum rule using a classical measure (Markov sum rule)

*Observables and operators cannot identify labels.

Quantum Realizations of Causal Sets

Causal sets are discrete and non-local, but not fully quantum.

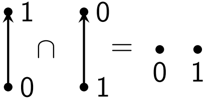
- CSG and MCMC use a classical probability measure
- No quantum operators!
- Some work* suggests placing spins on edges
- Recent work† explores complex measure spaces
- Should causal structure be definite in a deep quantum regime? (No)
- If it is not, how does it emerge in the large N limit? (Stabilized locally by matter)
- Are partial orders fundamental in the quantum setting? (No)

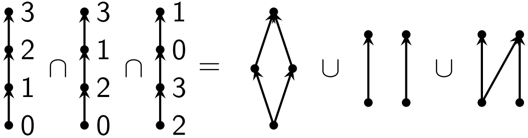
*Rideout & Sorkin '99; Solza '13
† Surya & Zalel '20

Concrete Examples: Partial Orders from Total Orders

Szpilrajn Order-Extension Principle: N-element partial orders are intersections of up to $N!$ N-element total orders

Proposal: "quantum" partial orders are superpositions (linear combinations) of total orders





Theorem: A quantum measure over all partial orders can be written in terms of a quantum measure over all total orders.

Partial Orders as Basis States

We can decompose states as follows:

$$|\Psi^n\rangle = \sum_{C_n^i \in \Omega_n} |\alpha(C_n^i)| e^{iS(C_n^i)/\hbar} |C_n^i\rangle$$

where $|C_n^i\rangle$ is the (quantum) state corresponding to a vertex in the growth tree.

We now wish to describe a quantum Markov process which describes how the wavefunction evolves. External time is simply N, internal time is the length of the longest chain L.

Quantum Random Walks

The process we just described is a discrete-time quantum random walk over the decision tree graph.

- A *walker* navigates a product Hilbert space $\mathcal{H} = \mathcal{H}_v \otimes \mathcal{H}_c$: vertex space crossed with the coin space.
- The *coin* will determine how much of the parent wavefunction moves to a given child
- In graphs, the coin dimension depends on the maximum degree (valency): the decision tree has at most 2^n children at depth n because a new element chooses yes/no (timelike/spacelike) for each existing element (or a superposition!)
- This indicates we can assign each element spin-1/2 degrees of freedom to implement a QRW

The Coin Hilbert Space

In general, each state has its own coin:

$$|C_n^i\rangle \in \mathbb{C}^{2^n}$$

but this leads to an overcounting for QSG.

- Isomorphic causal sets are multiply represented
- General covariance: isomorphic transitions should be equal
- Transitivity: combinatoric factor for ways to connect
- Connection "order" should not play a role

We use N Majorana fermions per unlabeled C_N, and $N|\Omega_N|$ of them in the N^{th} layer

Can we use a smaller coin Hilbert space?

Total Orders vs Partial Orders

Fundamental objects are permutations of labeled events, $\pi = \{1, 2, 3, \ldots, N\}$.

- A total order is already a 1D causal set
- Spacelike events only exist in superposition, $a \prec b \wedge b \prec a$
- Growth amounts to insertion of the n^{th} entry
- *Quantum N-order*: complete graph of N vertex spins, and $N(N-1)/2$ edge spins encode the coin
- Natural division into "causal/edge" spins and "matter/vertex" spins
- Far fewer vertex states ($N!$ vs $2^{N^2/4}$) and smaller coin Hilbert space

Proposal: Quantization of Causal Order

Theorem: A superposition over all N-element partial orders can be described by a superposition over all N-element total orders.

- Order is an exclusive quantum number q_i
- $\hat{q}_i^\dagger|\varnothing\rangle = |q_i\rangle$ and $\hat{q}_i|q_i\rangle = |\varnothing\rangle$ create/annihilate
- Total order: $\hat{q}_j^\dagger \hat{q}_i^\dagger|\varnothing\rangle = |q_j q_i\rangle \Rightarrow q_i \prec q_j$
- Interactions: $|c_{ij}\rangle = \theta(i-j)|0\rangle + \theta(j-i)|1\rangle$
- Permutation: $\hat{\pi} = q_a \rightarrow q_b \rightarrow \cdots$, $\{a, b, c, \ldots\}$ are a permutation of $\{1, 2, 3, \ldots, N\}$
- Superposition over Order: $|\Psi_n\rangle = \sum_{i=1}^{N!} p_i \hat{\pi}_i|\varnothing\rangle$
- Skew-symmetric Hamiltonian provides a Wick rotation
- Entropically dominated by random N-orders

Superpositions of Orders

$\hat{\pi}_1 = \hat{q}_4^\dagger \hat{q}_3^\dagger \hat{q}_2^\dagger \hat{q}_1^\dagger$, $\hat{\pi}_2 = \hat{q}_4^\dagger \hat{q}_2^\dagger \hat{q}_3^\dagger \hat{q}_1^\dagger$, and $|\psi\rangle = (\hat{\pi}_1 + \hat{\pi}_2)|\varnothing\rangle$

The interaction becomes $|c_{23}\rangle + |c_{32}\rangle = |+\rangle$ and the elements anti-commute when spacelike: $|\psi\rangle = \hat{q}_4^\dagger \{\hat{q}_3^\dagger, \hat{q}_2^\dagger\} \hat{q}_1^\dagger |\varnothing\rangle$

Add a third total order: $\hat{\pi}_3 = \hat{q}_2^\dagger \hat{q}_1^\dagger \hat{q}_4^\dagger \hat{q}_3^\dagger$ so $|\psi\rangle = (\hat{\pi}_1 + \hat{\pi}_2 + \hat{\pi}_3)|\varnothing\rangle$

The partial orders in $|\Psi\rangle$ are linear combinations of intersections of total orders

Spin-graph represented by an upper-triangular matrix of Majorana spins

Causality operator $\hat{C}$ measures these spins in the z-basis

The Quantum Markov Chain

Dynamics in a QRW are governed by a (semi-)unitary *growth operator* $\hat{V} = \hat{T} \cdot \hat{D}$.

$\hat{T}$ is the *transition operator* and $\hat{D}$ is the *coin operator* and the density matrix is

$$\rho_N = \hat{V}^{N-1}|\Psi_1\rangle\langle\Psi_1|(\hat{V}^\dagger)^{N-1}$$

We seek a form $V_{ij} = A(O_i, O_j)T_{ij}$, so that $\hat{V}$ encodes transition amplitudes.

The coin operator will update the internal degrees of freedom according to the current state, with the form

$$\hat{D} = e^{i\hat{H}/\hbar}$$

Unitary Growth

The $\hat{T}$ operator is block-diagonal:

$$\hat{T} = \begin{bmatrix} \hat{T}_1 & 0 & 0 & \cdots & 0 \\ 0 & \hat{T}_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \dots & \hat{T}_N \end{bmatrix}$$

where $\hat{T}_n \in \mathbb{C}^{(n!)\times(n+1)!}$ encodes transitions between layers n and $n+1$. They have entries

$$(\hat{T}_n)_{ij} = \frac{\mathrm{Aut}(O_{n+1})}{\mathrm{Aut}(O_n)} \delta(O_n^j \subset O_{n+1}^i)$$

where $\mathrm{Aut}(X)$ is the number of natural labelings (automorphisms).

Hamiltonian Dynamics

We can similarly use a reduced form of the coin operator:

$$\hat{D}_n = e^{i\hat{H}_n/\hbar}$$

so that $\langle \Psi^{n+1} | \hat{T} \cdot \hat{D} | \Psi^n \rangle$ gives the transition amplitudes between adjacent layers, and $\langle \Psi^N | \hat{V}^N | \varnothing \rangle$ gives the quantum measure over total orders.

The Hamiltonian $\hat{H}_n$ will be a function of the coin degrees of freedom – $n(n+1)/2$ spins. Obvious operators to act on them are the Pauli matrices:

$$\hat{H}_n = \frac{\Lambda^2}{N}\left(\hat{C}_n + J_{cs}\hat{\sigma}^z \otimes \hat{C}_n \otimes \hat{\sigma}^z + g_0 \hat{g}_n \otimes \hat{\sigma}^x\right)$$

This suggests for causal sets we may study an Ising-type action

$$\hat{H}_n = -J_1 \sum_{i<j}^{n} \hat{\sigma}_i^z \hat{\sigma}_j^z - J_2 \sum_{i=1}^{n} \hat{\sigma}_i^z \rightarrow S(C_n^i) = -J_1 \sum_{i<j<k<m}^{n} R_{ij} \veebar R_{km} - J_2 N$$

where J_1 corresponds to G and J_2 corresponds to Λ^2 in the context of asymptotic safety.